Statistical Delay Control of Opportunistic Links in Cognitive Radio Networks

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***Abstract*—Cognitive radio (CR) technology has been considered promising to enhance spectrum efficiency via opportunistic transmission at link level. To make CR useful, networking CRs to form a cognitive radio network (CRN) is able to support end-to-end transmission from CR source to CR destination. However, the opportunistic nature of CR link for interference avoidance to primary users degrades the quality-of-service (QoS) of end-to-end CR transmission and challenges the CRN toward a completely successful operation. Through queueing analysis, we propose a statistical control mechanism to deal with such opportunistic links in CRN by cooperative relaying the same packet flows into several opportunistic paths simultaneously. The availability and reliability of redundant transmission over the end-to-end paths in the same group is enhanced. By maximizing the number of groups with bounded statistical delay, the spectrum efficiency is enhanced. This optimized grouping problem is mathematically equivalent to the bin covering problem with NP-hard complexity. By the proposed Round-Robin algorithm, simulation results show that the optimal performance can be achieved in the case that the statistical availabilities of all opportunistic links are the same. This work therefore provides an essential viewpoint via cooperative relay among CRs in CRN, the QoS (i.e., average delay) can be guaranteed and spectrum can be efficiently utilized.**

Index Terms*—Availability, bin covering optimization, cognitive radio network, cooperative relay, delay model, dynamic spectrum access, opportunistic link

I. Introduction

Due to the underutilization of the existing scarce radio spectrum, dynamic spectrum access (DSA) for cognitive radios (CRs) is a promising solution receiving dramatic attentions [1]. In conventional development of CR technology, one common approach is the opportunistic spectrum access (OSA) [2], which allows CRs to seek and exploit the temporary spectrum holes without causing interference to primary system users (PUs). Under this consideration, the behavior of opportunistic transmission at a CR link introduces unpreventable delay. To network CRs as a cognitive radio network (CRN), Quality of Service (QoS) provisioning for end-to-end packet transmissions is an essential must. However, the aggregated delay from opportunistic links along a transmission path makes packet level QoS guarantee as a challenging issue.

Since the availability of radio spectrum at an opportunistic link depends mainly on the activity of PUs in vicinity, it is infeasible to provide deterministic QoS guarantees (i.e., a zero probability of violating the QoS constraint) for end-to-end CR transmission along a path consisting of multiple opportunistic links (denoted as an opportunistic path). Throughout this paper, the QoS guarantee is in statistical sense, that is, the QoS violation probability is ensured below a required value.

Cooperative relay among CRs emerges as a solution to exploit user diversity and provides dramatic gains in reliability and capacity. Spatial diversity gain of a *single communication link* offered by beam-forming at a CR/cooperative relay was discussed in [3]–[5]. The authors in [6] proposed a mechanism to select the best relay from the potential CR relay group for the throughput maximization at a CR link under the QoS constraint of PUs. [7] explored the problem in a *network* perspective, where a cooperative scheduling algorithm was proposed to select a relay opportunistic path with the lowest packet dropping probability for end-to-end CR transmissions.

To enable packet level QoS guarantees in CRN, this paper introduces redundant transmissions through multiple opportunistic paths to increases the reliability and availability. Specifically, same packet flows are simultaneously delivered over opportunistic paths in the same group to ensure the statistical delay. Since the larger number of groups leads higher aggregated throughput at the same time, this paper tries to maximize the number of groups while ensuring the statistical delay of each redundant transmission over the same group. This optimization problem is proved as the well known bin covering optimization with NP-hard complexity. By the proposed Round-Robin algorithm, simulation results show that the optimal performance can be achieved in the case that the statistical availabilities of all opportunistic links are the same. This work therefore provides an essential viewpoint that via cooperative relay among CRs in CRN, the QoS (i.e., average delay) can be guaranteed and spectrum can be efficiently utilized.

The remainder of this paper is organized as follows. In section II, system model is presented. Section III formulates our problem and proposes an algorithm. Section IV presents the performance evaluation results. Finally, section V makes a conclusion.

II. System Model

A. Network Topology

We consider a network coexisted by CRs and PUs, where a CRN is existed consisting of a source CR (denoted as node n_S), a destination CR (denoted as n_D), and several relay CRs (denoted as n_Rs) that can help relay packet flows from the n_S to n_D. Assume that CRN is a slotted system and the n_S and the n_D are synchronous with time interval unit Δt, which consists

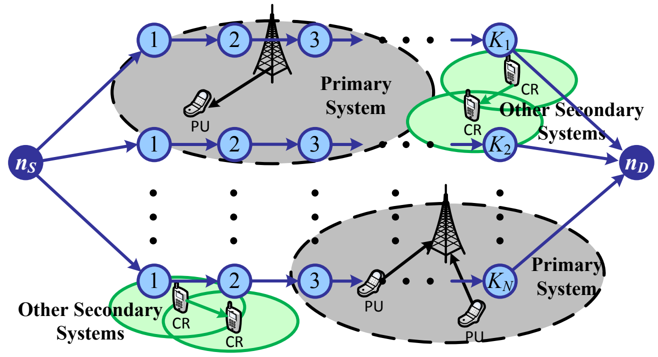

Fig. 1. Network Topology of CRRN

of spectrum sensing time and spectrum access time for data transmission. In the spectrum sensing period, the transmitter of the CR link shall perform channel sensing to identify whether the link is occupied by the PU. If it does, the subsequent data transmission period is unavailable for packet transmissions. We assume that the n_S, the n_D, and the n_Rs are mobile but stationary during each D time intervals, that is, the network topology varys per $D\Delta t$. Fig. 1 shows a stationary topology during some $D\Delta t$.

Suppose that there are N possible opportunistic paths between the n_S and the n_D. The set of the N opportunistic paths is denoted as $P = \{p_1, p_2, \ldots, p_N\}$, where the ith opportunistic path p_i consists of K_i links, for $i = i, 2, \ldots, N$. The links number of all opportunistic paths are labeled by $\kappa = \{K_1, K_2, \ldots, K_N\}$. We further assume that K_i is a geometric random variable with parameter α and denoted as $G_{1,\alpha}$ (This notation is explained by the fact that a geometric random variable is a special case of a negative binomial random variable denoted by $G_{n,\alpha}$ [8]), for $i = i, 2, \ldots, N$.

B. Opportunistic Links

To avoid causing destructive interference to PUs, CR links are mandated to exploit temporary spectrum holes for data transmission [1], [9]. DSA can effectively fetch such opportunities after successful spectrum sensing. Thus, the transmission opportunity on a CR link can be mathematically modeled as a two-state continuous-time Markov chain with the available state (the PU does not occupy the link) and the unavailable state (the PU occupies the link) as the PU is in general not slotted and could occupy any links at any time instants [10]. The state transition diagram of the kth opportunistic link of the ith opportunistic path in the sth time interval is shown in Fig. 2. We could formulate the available probability of the kth opportunistic link in the ith opportunistic path in the sth time interval by following Bernoulli process as

$$\pi_{i,k,s} = \frac{P_{01}^{i,k,s}}{P_{01}^{i,k,s} + P_{10}^{i,k,s}}, \quad (1)$$

where 0 is unavailable state, 1 is available state, $P_{01}^{i,k,s}$ is transition probability from state 0 to state 1, and $P_{10}^{i,k,s}$ is

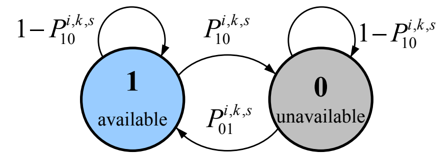

Fig. 2. State transition diagram of kth link of ith opportunistic path in sth time interval

transition probability from state 1 to state 0. When considering CRN for a long term period, we can reasonably assume that $P_{01}^{i,k,s}$ and $P_{01}^{i,k,s}$ are the same for all possible i, k, and s. Thus, $\pi_{i,k,s}$ is the same for all possible i, k, and s. In the following, $P_{01}^{i,k,s}$, $P_{10}^{i,k,s}$, and $\pi_{i,k,s}$ are respectively abbreviated as P_{01}, P_{10}, and π for simplification.

C. Traffic Model

The traffic considered in this paper is the traditional packet session, which comprise series of packets with Poisson arrival rate λ and fixed service time Δt. The behavior of the traffic at each opportunistic link can be modeled as a M/D/1/∞/FCFS queueing model with available probability π. If an opportunistic link is unavailable in a time interval, the transmitted packet would be back-off and try to transmit again in the next time slot until the opportunistic link is available to transmit. Thus, the M/D/1/∞/FCFS queueing model with available probability π is equivalent to a M/Geo/1/∞/FCFS queueing model, where service time is geometrically distributed with parameter π (denoted as $G_{1,\pi\Delta t}$).

Since the network topology is dynamic per $D\Delta t$, we set the QoS requirement (i.e., delay constraint) as $D\Delta t$. That is, if a packet cannot be transmitted from the n_S to the n_D within $D\Delta t$, the packet would be dropped.

D. Delay Model

1) Delay of an Opportunistic Link: Suppose that packets of the n_S are served in the order they arrive in the first opportunistic link and that X_i is the service time of the ith arrival packet. We assume that the random variables $(X_1, X_2, \ldots)$ of $G_{1,\pi\Delta t}$ are identically distributed, mutually independent, and independent of the inter arrival times. Thus, the mean of the random variable X is

$$E[X] = \Delta t \sum_{n=1}^{\infty} n\pi(1-\pi)^{n-1} = \frac{\Delta t}{\pi}. \quad (2)$$

The mean of the random variable X^2 is

$$\begin{aligned} E[X^2] &= V[X] + E[X]^2 \\ &= \frac{\Delta t^2(1-\pi)}{\pi^2} + \left(\frac{\Delta t}{\pi}\right)^2 = \Delta t^2 \frac{(2-\pi)}{\pi^2}, \end{aligned} \quad (3)$$

where $V[X]$ is the variance of random variable X. For M/GI/1 queueing system, we can apply *Pollaczek-Khinchin* (P-K) formula [11] to obtain the average waiting time W of the packets in an opportunistic link as

$$W = \frac{\lambda E[X^2]}{2(1-\lambda E[X])} = \frac{\lambda \Delta t^2 (2-\pi)}{2\pi(\pi - \lambda\Delta t)}. \quad (4)$$

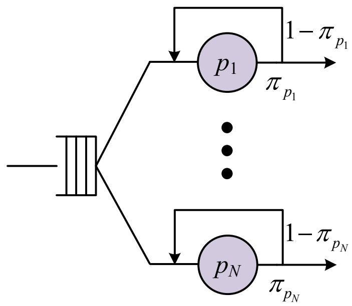

Fig. 3. Queueing node of all opportunistic paths

Thus, the statistical delay consisting of serving time and waiting time in an opportunistic link can be derived as

$$D^{\text{link}} = \frac{\Delta t}{\pi} + \frac{\lambda \Delta t^2 (2-\pi)}{2\pi(\pi - \lambda\Delta t)} = \frac{\Delta t(2-\lambda\Delta t)}{2(\pi-\lambda\Delta t)} \tag{5}$$

2) Delay of an Opportunistic Path: The queueing model of the ith opportunistic path consisting of K_i opportunistic links will be derived as a M/Geo/1/∞/FCFS queueing system, where K_i is a geometric distribution with parameter α. Obviously, the service time of the opportunistic path is the geometric sum of geometric random variables and it can be derived as a geometric random variable with parameter $\alpha\pi\Delta t$. By denote $\pi_{p_i} = \alpha\pi$ as the available of an opportunistic path, the service time random variable can be represented as $G_{1,\pi_{p_i}\Delta t}$. We assume that the random variables of paths are identically distributed, mutually independent, and independent of the inter arrival times. By adopting (5), the average delay of the packets in the ith opportunistic path D_i^{path} is

$$D_i^{\text{path}} = \frac{\Delta t(2-\lambda\Delta t)}{2(\pi_{p_i}-\lambda\Delta t)}, \tag{6}$$

for $i = 1, 2, \ldots, N$.

3) Delay of a Group of Opportunistic Paths with Redundant Transmissions: Please note that the queueing model of the ith opportunistic path (M/Geo/1/∞/FCFS) can be represented as M/D/1/∞/FCFS with arrival rate λ, service time Δt, and available probability $\pi_{p_i} = \alpha\pi$, for $i = 1, 2, \ldots, N$. Fig. 3 shows us an example. If n opportunistic paths are merged as a group (i.e., transmitted with the same session packet flows), similar to (6), the average delay of the packet flows via the group can be derived as

$$D^{\text{group}} = \frac{\Delta t(2-\lambda\Delta t)}{2(\pi_g-\lambda\Delta t)}, \tag{7}$$

where π_g is the available probability of the group and is obtained by

$$\pi_g = 1 - \prod_{i=1}^{n}(1-\pi_{p_i}). \tag{8}$$

From (8), we observe that available probability of a group π_g is larger than the availability probability of any opportunistic path in this group (i.e., π_{p_i} for $i = 1, 2, \ldots, n$), the average delay of the group is smaller and can bes regarded as the cooperative delay gain.

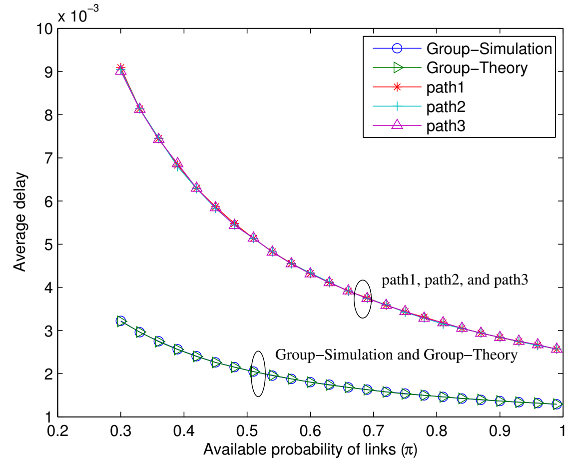

Fig. 4. Average delay of three opportunistic paths with the same packet flows

We conduct a simulation to verify the correctness of (7) with parameter settings $\lambda = 10$, $\Delta t = 0.001s$, and $\alpha = 0.4$. In the simulation, we always merge three opportunistic paths as a group. As shown in Fig. 4, we obverse that the results of the analytical model are consistent with the simulation results.

Since D^{group} is a random variable of the delay value of the group, we can guarantee that $E[D^{\text{group}}]$ is lower than τ by employing the Markov inequality [12]. That is,Marin

$$\Pr(D^{\text{group}} \geq a) \leq \frac{E[D^{\text{group}}]}{a} \leq \frac{\tau}{a}, \tag{9}$$

where $a > 0$. In other words, we can provide the statistical delay guarantee by Markov inequality.

In order to guarantee that $E[D^{\text{group}}]$ is lower than τ, we must guarantee that the available probability of the group is higher than η. We can obtain η in the following equations.

$$D^{\text{group}} \leq \tau. \tag{10}$$

$$\pi_g \geq \eta = \frac{\Delta t(2-\lambda\Delta)}{2\tau} + \lambda\Delta t. \tag{11}$$

Thus, the dropping probability (i.e., the probability of violating delay requirement, $D\Delta t$) of a packet can be lower than $\frac{\tau}{D\Delta t}$ and is shown by

$$\Pr(D^{\text{group}} \geq D\Delta t) \leq \frac{\tau}{D\Delta t}. \tag{12}$$

III. Problem Formulation and Algorithms

A. Problem Formulation

This section proposes an algorithm to maximize the number of groups while guaranteeing the statistical delay of each redundant transmission over the same group. Please note that a set $P = \{p_1, p_2, \ldots, p_N\}$ of opportunistic paths exist in the CRN and each opportunistic path p_i corresponds to available probability π_{p_i}. The set P can be partitioned into groups

$G_1, G_2, \ldots, G_M$. Similar to (8), the available probability of each G_j is defined as

$$\pi_{G_j} = 1 - \prod_{p_i \in G_j} (1 - \pi_{p_i}), \text{ for } j = 1, 2, \ldots, M \tag{13}$$

In order to maximize system utilization (i.e., number of session packet flows assigned) and maximize aggregated bandwidth, we try to maximize the number of group M such that $\pi_{G_j} \geq \eta$ for all $j = 1, \ldots, M$. We name this maximization problem as MAXG problem and formulate it as follows.

MAXG Given a set $P = \{p_1, p_2, \ldots, p_N\}$ of opportunistic paths, an available probability $\pi_{p_i} \in [0, 1]$ for each opportunistic path p_i (where $i = 1, \ldots, N$), and an available probability threshold $\eta \geq 0$. Let $G_j \subset P$ and $\pi_{G_j} = 1 - \prod_{p_i \in G_j} (1 - \pi_{p_i})$, for $j = 1, \ldots, M$.

Maximize M

Subject to

$$\pi_{G_j} \geq \eta, \text{ for } j = 1, 2, \ldots, M \tag{14}$$

$$G_j \cap G_k = \emptyset, \text{ for } j \neq k \text{ and } j, k = 1, \ldots, M \tag{15}$$

$$G_1 \cup G_2 \cup \cdots \cup G_M = P, \tag{16}$$

We apply a polynomial time transformation in available probability of all opportunistic paths and threshold calculations, that is $\hat{\pi}_{p_i} = \log(\frac{1}{1-\pi_{p_i}})$ for $i = 1, \ldots, N$ and $\hat{\eta} = \log(\frac{1}{1-\eta})$. In this case, the corresponding function to calculate available probability of a group is defined as

$$\hat{\pi}_{G_j} = \sum_{p_i \in G_j} \hat{\pi}_{p_i}. \tag{17}$$

The constraint (14) is modified as

$$\hat{\pi}_{G_j} \geq \hat{\eta}, \text{ for } j = 1, 2, \ldots, M. \tag{18}$$

After the above polynomial time transformation, it is easy to observe that MAXG problem is equivalent to the bin covering problem [13], where one must pack the items into bins in such a way as to maximize the number of bins used, subject to the constraint that each bin be filled to at least a given bin size. The threshold $\hat{\eta}$ and available probability $\hat{\pi}_{p_i}$, for $i = 1, 2, \ldots, N$, are analogous to bin and item sizes, respectively. We now prove that MAXG problem is an NP-hard problem, based on a polynomial reduction from bin covering, which is known to be strongly NP-hard [14].

Bin covering Given a finite set U of items, a size $0 \leq s_{u_i} \leq 1$ for each $u_i \in U$ ($i = 1, \ldots, N$), a positive threshold T. What is the maximum number M such that U can be partitioned into sets $X_1, X_2, \ldots, X_M$, where each set has total size

$$s_{X_j} \equiv \sum_{u_i \in X_j} s_{u_i} \geq T, \tag{19}$$

and hence can fill a bin to least this threshold.

Theorem 1. *MAXG is a NP-hard problem.*

Proof: We consider the following transformation. Suppose that we find a maximum number of sets for bin covering as M^* and create M^* groups with same threshold for the MAXG as

$$\eta = 1 - e^{-T}. \tag{20}$$

Note that it satisfies the constraint that $0 \leq \eta \leq 1$. We let $|P| = |U|$ to create $|P|$ opportunistic paths for the MAXG. Each opportunistic path in P corresponds to one item in U, where we set the available probability of an opportunistic path as

$$\pi_{p_i} = 1 - e^{-s_{p_i}}, \text{ for } i = 1, \ldots, N \tag{21}$$

which satisfies the constraint that $0 \leq \pi_{p_i} \leq 1$, for $i = 1, 2, \ldots, N$. In the following, we show how the constraint (19) can be transformed as a valid constraint for MAXG.

$$\begin{aligned}
&\sum_{u_i \in X_j} -s_{u_i} \leq -T \\
&e^{\sum_{u_i \in X_j} -s_{u_i}} \leq e^{-T} \\
&\prod_{u_i \in X_j} e^{-s_{u_i}} = \prod_{p_i \in G_j} e^{-s_{p_i}} \leq e^{-T} \\
&\prod_{u_i \in G_j} (1 - \pi_{p_i}) \leq e^{-T} = 1 - \eta \\
&1 - \prod_{u_i \in G_j} (1 - \pi_{p_i}) = \pi_{G_j} \geq \eta
\end{aligned} \tag{22}$$

Obviously, (22) equals to constraint (14) from (13). Since find the maximum number of sets M^* in bin covering is a NP-hard problem, it is clear that find the maximum number of groups M^* in MAXG is last least NP-hard. ■

B. The Algorithms

In our system model, π_{p_i} is the same as π_{p_j}, for $i \neq j$ and $i, j = 1, 2, \ldots, N$. This is a specific case of the bin covering optimization. We propose a Round Robin algorithm to solve this specific case which is as follows:

Algorithm 1 ROUND ROBIN (RR)

```
1: m = 0 (Initialize m)
2: π_temp = 1 (Initialize π_temp)
3: for i = 1 to N do
4:     if π_{p_i} > η then
5:         m = m + 1
6:     else
7:         π_temp = π_temp · (1 − π_{p_i})
8:         if π_temp ≤ 1 − η then
9:             m = m + 1
10:            π_temp = 1
```

The current number of groups and current availability probability of the current group is denoted as m and π_{temp}. After initialization of m and π_{temp} at lines 1-2, lines 3-10 sequently group path $p_i \in P$ to satisfy constraint (14) in a round robin fashion. At lines 4-5, an ungrouped opportunistic path with availability probability larger than threshold makes current group "full" and m is increased by one. $O(1)$ arithmetic

TABLE I
PARAMETER SETUPS IN SIMULATION

Parameter	Description	Value
λ	Arrival rate	10
Δt	A time interval	$0.001s$
N	Number of opportunistic paths	10
α	Existing probability of opportunistic links	0.4
P_{10}	Transition probability from 1 to 0	0.0 to 1.0
P_{01}	Transition probability from 0 to 1	0.1
τ	Average delay constraint	$3\Delta t$
D	Number of time intervals which CRs are fixed position	300

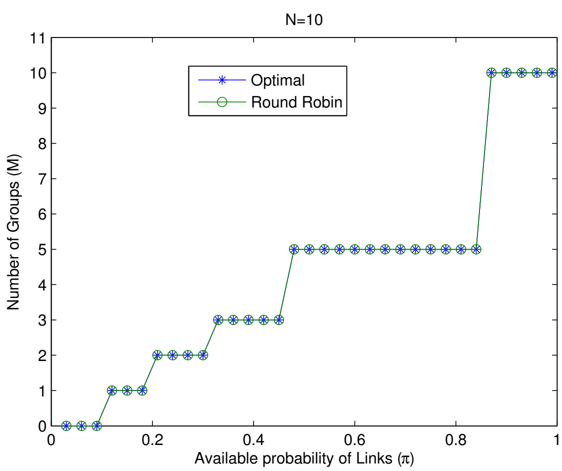

Fig. 5. Number of groups (M) v.s available probability of opportunistic links (π)

operations are respectively needed to find whether the available probability of each opportunistic path exceeds the threshold and to increase the number of group.

If the availability probability does not exceed, line 7 updates the availability probability of current group after adding the current path, which takes $O(1)$ arithmetic operations. At lines 8-10, if adding current path to current group makes availability probability of current group larger than threshold, m is increased by one. As a result, the n_S only needs $O(N)$ arithmetic operations to complete RR algorithm.

IV. PERFORMANCE EVALUATION

The parameters we set are shown in Table I. We randomly generate the matrix κ which $K_i \in \kappa$ follows $G_{1,\alpha}$, for $i = 1, 2, \ldots, N$. If we attempt to constrain the average delay to be lower than a given τ, we must guarantee the available probability of each group higher than η (i.e., we can obtain $\eta = 0.3417$ from (7). Thus, we can provide statistical delay guarantee (i.e., dropping probability is lower than 0.01) by Markov inequality (9). From the Fig. 5, the simulation results show that the Round Robin algorithm achieves the optimal performance of the MAXG. When π is higher than 0.9, the average delay of each opportunistic path is lower than τ. Thus, there are N (i.e., 10) groups. We find that n_S must merge more opportunistic paths as a group to transmit packet flows such the requirement of average delay can be achieved in the lower π (i.e., available probability of opportunistic links).

V. CONCLUSION

This paper proposes a statistical control mechanism which chooses multiple opportunistic paths (each opportunistic path consists of opportunistic links) as a group to transmit packet flows such that the statistical delay of this group is restricted to be lower than a target threshold. We further propose an algorithm to partition all possible opportunistic paths into several groups so that the average delay of packet flows in each group is constrained and the maximum number of groups is identified. The partition optimization problem is proved to be the well known bin covering optimization with NP complexity. A specific case that the statistical availability of all opportunistic links are the same is investigated to understand the intrinsic behavior of our problem. By the proposed RR algorithm, simulation results show that the maximum number of groups can be achieved in the case, thereby providing an essential viewpoint for further studies on general CRN topology.

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